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W. S. Hall, *Descriptive Geometry*, New York, 1903.
S. E. Warren, *Elements of Descriptive Geometry*, New York, 1905.
J. B. Millar, *Elements of Descriptive Geometry*, New York, 1905.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

34 [*1917*, 134, 341; *1920*, 114, 301, 405]. Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

I. REMARKS BY A. A. BENNETT, Baltimore, Md.

For the functional equation of Simpson's Rule, the origin is a point specially characterized. If smoothness at the origin be not required, solutions which are continuous and contain an infinite number of parameters may be obtained which present irregularities at this point. In particular the following one-parameter family of analytic functions with an essential singularity at the origin satisfy the equation:

$$x^a \sin (b \log x - c),$$

where a and b are constants satisfying certain transcendental equations and where c is an arbitrary parameter. This solution may furthermore be added to any other solution to yield a solution.

For the equation

$$\int_0^h f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

if f is differentiable,[1] and $f(0) = 0$, becomes, after differentiation

$$6f(x) = [4f(x/2) + f(x)] + x[2f'(x/2) + f'(x)].$$

Testing $f(x) \equiv x^a \sin (b \log x - c)$ we have

$$f'(x) = x^{a-1}[a \sin (b \log x - c) + b \cos (b \log x - c)].$$

Substituting throughout this gives

$$\begin{aligned} 6x^a \sin (b \log x - c) = (x^a/2^{a-2}) \sin (b \log x/2 - c) + x^a \sin (b \log x - c) \\ + (x^a/2^{a-2})[a \sin (b \log x/2 - c) + b \cos (b \log x/2 - c)] \\ + x^a[a \sin (b \log x - c) + b \cos (b \log x - c)]. \end{aligned}$$

Collecting terms, and canceling x^a, this yields

$$\begin{aligned} (5 - a) \sin (b \log x - c) - b \cos (b \log x - c) \\ = (1/2^{a-2})[(a + 1) \sin (b \log x/2 - c) + b \cos (b \log x/2 - c)]. \end{aligned}$$

Writing $b \log x/2 - c$ as $(b \log x - c) - b \log 2$ and expanding the terms in which this occurs, then equating coefficients of $\sin (b \log x - c)$ and of $\cos (b \log x - c)$, one obtains

$$\begin{cases} (5 - a)\cdot 2^{a-2} = (a + 1) \cos (b \log 2) + b \sin (b \log 2) \\ b\cdot 2^{a-2} = - b \cos (b \log 2) + (a + 1) \sin (b \log 2). \end{cases}$$

These equations are satisfied by an infinite set of values of a, b.[2]

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[1] The assumption of differentiability is not essential. Direct substitution of the function $x^a \sin (b \log x - c)$ in the given equation leads to a pair of conditions on a, b equivalent to those derived in the text.

[2] See the remarks by the Editor, immediately following the present article.

In order to exhibit the special rôle of the point $x = 0$, we shall obtain from the given equation another not involving this point; and then examine the nature of solutions of the new equation and under what conditions the old equation can be derived from the new. We assume first that $f(x)$ is continuous, and start with the given equation.

$$\int_0^x f(t)dt = \frac{x}{6}[f(0) + 4f(x/2) + f(x)]. \tag{1}$$

This is a special case of

$$\int_0^x f(t)dt = \frac{x}{6}[C + 4f(x/2) + f(x)]. \tag{2}$$

In (2), give x the constant value $a \neq 0$; thus

$$\int_0^a f(t)dt = \frac{a}{6}[C + 4f(a/2) + f(a)]; \tag{3}$$

and solve for C, thus

$$C = \frac{6}{a}\int_0^a f(t)dt - 4f(a/2) - f(a). \tag{4}$$

Substitute in (2),

$$\int_0^x f(t)dt = \frac{x}{6}\left[\frac{6}{a}\int_0^a f(t)dt - 4f(a/2) - f(a) + 4f(x/2) + f(x)\right]. \tag{5}$$

In (5), replace x by the constant $2a$,

$$\int_0^{2a} f(t)dt = \frac{a}{3}\left[\frac{6}{a}\int_0^a f(t)dt - 4f(a/2) + 3f(a) + f(2a)\right]. \tag{6}$$

Subtract (5),

$$\begin{aligned}\int_x^{2a} f(t)dt = \left(2 - \frac{x}{a}\right)\int_0^a f(t)dt - \tfrac{2}{3}(2a - x)f(a/2) + \tfrac{1}{6}(6a + x)f(a) \\ + \tfrac{1}{3}af(2a) - \tfrac{2}{3}xf(x/2) - \tfrac{1}{6}xf(x).\end{aligned} \tag{7}$$

From (6), subtract $\int_0^a f(t)dt$, giving

$$\int_a^{2a} f(t)dt = \int_0^a f(t)dt - \tfrac{4}{3}af(a/2) + af(a) + \tfrac{1}{3}af(2a) \tag{8}$$

or

$$\int_0^a f(t)dt = \int_a^{2a} f(t)dt + \tfrac{4}{3}af(a/2) - af(a) - \tfrac{1}{3}af(2a). \tag{9}$$

Substitute in (7), giving the equation

$$\begin{aligned}\int_x^{2a} f(t)dt = \left(2 - \frac{x}{a}\right)\int_a^{2a} f(t)dt + \tfrac{2}{3}(2a - x)f(a/2) - \tfrac{1}{6}(6a - 7x)f(a) \\ - \tfrac{1}{3}(a - x)f(2a) - \tfrac{2}{3}xf(x/2) - \tfrac{1}{6}xf(x).\end{aligned} \tag{10}$$

Solving for $4xf(x/2)$, we have

$$\begin{aligned}4xf(x/2) = -xf(x) - 6\int_x^{2a} f(t)dt + 6\left(2 - \frac{x}{a}\right)\int_a^{2a} f(t)dt \\ + (8a - 4x)f(a/2) - (6a - 7x)f(a) - (2a - 2x)f(2a).\end{aligned} \tag{1'}$$

For the new equation (1′) we have the following

THEOREM: *Given any positive number, a, and any real one-valued function continuous in the interval $I : a \leqq x \leqq 2a$, the end points being included, and further given an arbitrary value, A,—there exists a unique solution of* (1) *coinciding with the given function in I, taking on the value A at a/2 and continuous for all real positive values of x.*

It is at once obvious that the value of f(1′) in $I : a \leqq x \leqq 2a$ and its value at $a/2$ determine from (1′) its value in $a/2 \leqq x \leqq a$; by substitution of $x = 2a$ in (1′) it appears that $f(x)$ is continuous in both directions at $x = a$. By the methods of integral equations the value of $f(x)$ is determined by (1′) in $2a \leqq x \leqq 4a$. By repetition of this process the definition of the function can be extended repeatedly to intervals on the left, each half as long as the preceding, and to intervals on the right, each twice as long as the preceding. It need not be true, however, that the values so obtained on the left approach any limit at all as x approaches zero.

In order to return from (1′) to (1) we make the following assumptions:

(I) $f(x)$ is continuous, $x > 0$, and satisfies (1′) for some value of the constant a.

(II) $f(x)$ is bounded in a neighborhood of $x = 0$.

(III) If $\int_0^x f(t)dt$ can be written in the form $\frac{x}{6}\left[C + 4f\left(\frac{x}{2}\right) + f(x)\right]$, then $f(0)$ exists and $C = f(0)$.

The condition (II) may be replaced by the two weaker conditions:

(II*a*) $\lim\limits_{x \to 0} xf(x) = 0$,

(II*b*) $\int_0^x f(t)dt$ converges for $x > 0$.

Condition (III) may be replaced by

(III*a*) $f(0)$ is defined and $f(x)$ is continuous at $x = 0$. If (III*a*) is assumed, (II) is superfluous.

Starting with

$$4xf(x/2) = -xf(x) - 6\int_x^{2a} f(t)dt + 6\left(2 - \frac{x}{a}\right)\int_a^{2a} f(t)dt$$
$$+ (8a - 4x)f(a/2) - (6a - 7x)f(a) - (2a - 2x)f(2a) \qquad (1')$$

we have, upon letting x approach 0,—thus utilizing condition (II), or (II*a*) and (II$_b$)—the following.

$$0 = -6\int_0^{2a} f(t)dt + 12\int_a^{2a} f(t)dt + 8af(a/2) - 6af(a) - 2af(2a),$$

whence we reobtain,

$$\int_0^a f(t)dt = \int_a^{2a} f(t)dt + \tfrac{4}{3}\,af(a/2) - af(a) - \tfrac{1}{3}\,af(2a). \qquad (9)$$

Eliminating $\int_a^{2a} f(t)dt$ between (9) and (1′) we have again

$$\int_x^{2a} f(t)dt = \left(2 - \frac{x}{a}\right)\int_0^a f(t)dt - \tfrac{2}{3}\,(2a - x)\,f(a/2) + \tfrac{1}{6}\,(6a + x)\,f(a)$$
$$+ \tfrac{1}{3}\,af(2a) - \tfrac{2}{3}\,x\,f(x/2) - \tfrac{1}{6}\,x\,f(x). \qquad (7)$$

In (7), letting x approach 0, we obtain

$$\int_0^{2a} f(t)dt = \frac{a}{3}\left[\frac{6}{a}\int_0^a f(t)dt - 4f(a/2) + 3f(a) + f(2a)\right]. \qquad (6)$$

Subtracting (7) from (6), we have,

$$\int_0^x f(t)dt = \frac{x}{6}\left[\frac{6}{a}\int_0^a f(t)dt - 4f(a/2) - f(a) + 4f(x/2) + f(x)\right], \qquad (5)$$

which is of the form

$$\int_0^x f(t)dt = \frac{x}{6}[C + 4f(x/2) + f(x)]. \qquad (2)$$

By employing condition (III), we return to

$$\int_0^x f(t)dt = \frac{x}{6}[f(0) + 4f(x/2) + f(x)], \qquad (1)$$

as desired.

Thus solutions of (1′) which satisfy (I), (II), (III) or (I), (II*a*), (II*b*), (III) or (I), (III*a*) will be solutions of (1). Solutions of (1′) include, however, functions not satisfying these further conditions bearing on the point $x = 0$ and therefore not serving as solutions of the original equation.

II. REMARKS BY THE EDITOR.

It will be of value to indicate the relationship of the new solutions found by Professor Bennett to the work which has previously appeared in connection with this question. A convenient method of approach is found in the fact, used in earlier replies, that $f(x) = x^n$ is a solution of the functional equation, in case n is a positive integer, if and only if

$$\frac{1}{n+1} - \frac{1}{6 \cdot 2^{n-2}} - \frac{1}{6} = 0. \tag{1}$$

We first extend our consideration to values of n, integral or fractional, real or complex. It is evident that for any n whose real part is positive, (1) still gives the condition that x^n be a solution. This may be written

$$\frac{n-5}{n+1} + \frac{1}{2^{n-2}} = 0;$$

or, letting

$$n - 2 = m,$$

$$\frac{m-3}{m+3} + 2^{-m} = 0. \tag{2}$$

In the equivalent statement

$$(m-3) + (m+3) \cdot 2^{-m} = 0$$

put

$$m = \alpha + i\beta,$$

and separate the real and pure imaginary parts of the resulting equation. Writing for brevity

$$\log 2 = l,$$

we have

$$(\alpha - 3) + 2^{-\alpha}[(\alpha + 3) \cos \beta l + \beta \sin \beta l] = 0, \tag{3}$$

$$\beta + 2^{-\alpha}[\beta \cos \beta l - (\alpha + 3) \sin \beta l] = 0. \tag{4}$$

Multiply (3) by $\sin \beta l$, (4) by $\cos \beta l$, and add:

$$(\alpha - 3) \sin \beta l + \beta \cos \beta l + 2^{-\alpha} \cdot \beta = 0. \tag{5}$$

Multiply (4) by 2^{α}, and subtract (5) from it:

$$\beta[2^{\alpha} - 2^{-\alpha}] - 2\alpha \sin \beta l = 0.$$

The latter equation may also be written

$$\beta \sinh \alpha l = \alpha \sin \beta l. \tag{6}$$

This equation can have no solutions for which both α and β differ from zero. In fact, if $\alpha \neq 0$, $\beta \neq 0$, (6) may be written

$$\frac{\sinh \alpha l}{\alpha l} = \frac{\sin \beta l}{\beta l};$$

but $(\sinh u)/u > 1$ and $(\sin u)/u < 1$, for all real $u \neq 0$, so that the equation cannot hold.

In case α or β vanishes, (6) holds. It remains to be seen when (3), (4) are satisfied. First let $\beta = 0$; then (4) is satisfied, and (3) reduces to (2), with α replacing m. This equation is readily seen to have the solutions $\alpha = -1, 0, 1$, corresponding to $n = 1, 2, 3$. It will be shown that there are no others. Writing the equation in the form

$$\frac{2^{\alpha/2}}{2^{-\alpha/2}} = \frac{3+\alpha}{3-\alpha},$$

we have

$$\frac{2^{\alpha/2} - 2^{-\alpha/2}}{2^{\alpha/2} + 2^{\alpha/2}} = \frac{\alpha}{3},$$

or

$$\tanh \tfrac{1}{2}\alpha l = \tfrac{1}{3}\alpha.$$

This may be written

$$\tanh u - ku = 0, \tag{7}$$

where

$$u = \tfrac{1}{2}\alpha l, \qquad k = \frac{2}{3l}.$$

Now the function $\tanh u - ku$ vanishes for $u = 0$; if it vanished for two positive values of u, then by Rolle's theorem, its first derivative would vanish for at least two positive values, and its second derivative for at least one. But its second derivative is $-2 \tanh u \operatorname{sech}^2 u$ and never vanishes for $u \neq 0$. Thus (7) has at most one positive solution, and similarly at most one negative solution.

Finally let $\alpha = 0$. Equations (4), (3) reduce to

$$3 \sin \beta l = \beta(1 + \cos \beta l),$$

$$\beta \sin \beta l = 3(1 - \cos \beta l);$$

and this pair of equations is equivalent to the single equation

$$3 \sin \tfrac{1}{2}\beta l = \beta \cos \tfrac{1}{2}\beta l.$$

Since it is clear that no value which makes $\cos \frac{1}{2}\beta l$ vanish can be a solution, the only admissible values of β are those for which

$$\tan \tfrac{1}{2}\beta l = \tfrac{1}{3}\beta, \tag{8}$$

which may be written

$$\tan u - ku = 0, \tag{9}$$

where

$$u = \tfrac{1}{2}\beta l, \qquad k = \frac{2}{3l}.$$

This is a type of equation which arises in many problems of analysis. Inspection of the graphs of $\tan u$ and ku leads at once to the well-known fact, which can readily be proved by rigorous analytic methods, that (8) has an infinite number of solutions, approximating more and more closely, as they increase numerically, to odd multiples of $\pi/2$.

We condense all these results into the statement: The function x^n, where n is a constant whose real part is positive, satisfies the proposed functional equation for the following values of n and no others:

(*a*) The real values $n = 1, 2, 3$.

(*b*) An infinite set of values of the form $n = 2 + \beta i$, where β is a solution[1] of (8).

Any of the second set of solutions leads to real solutions of the type considered by Professor Bennett. In fact, since

$$x^n = x^{m+2} = x^{\alpha+2+\beta i} = x^{\alpha+2} \cos (\beta \log x) + i x^{\alpha+2} \sin (\beta \log x),$$

for any n yielding a complex solution we have the two real solutions $x^{\alpha+2} \cos (\beta \log x)$ and $x^{\alpha+2} \sin (\beta \log x)$, whence also the more general solution $x^{\alpha+2} \sin (\beta \log x - c)$, which is a linear combination of them. Equations (3), (4) are equivalent to Professor Bennett's pair of equations for a, b with $a = \alpha + 2$, $b = \beta$. We have thus shown that Professor Bennett's formula gives a solution only (except for the trivial cases $b = 0$, $a = 1, 2, 3$) when $a = 2$, and that in this instance an infinite number of values of b will be possible. It is obvious that much more general solutions can be constructed, of the form $x^2F(x)$, where

$$F(x) = \Sigma[A_p \cos (\beta_p \log x) + B_p \sin (\beta_p \log x)],$$

proper precautions being taken to insure the validity of the series.

The importance of these new solutions lies in the following facts. The function $x^2 \sin (\beta \log x - c)$ possesses a continuous first derivative even at the origin. In previous editorial comment, it was stated that "it seems reasonable to adopt as a goal with reference to this question the proof that if the equation holds for some range of the variable h, the function $F(x)$ can only be a polynomial of degree $\leqq 3$, under restrictions as light as possible—*e.g.*, that $F(x)$ should be continuous and possess a stated number (as small as it can be made) of derivatives." The statement has now been proved, with the number of derivatives as great as *six*, by Professor Gillespie. Professor Bennett's example, together with the present comments, shows that the theorem is not true with the number of derivatives as small as *one*. Can the gap between these two results be bridged?

DISCUSSIONS.

In a previous number of the MONTHLY (*1920*, 53) Mr. W. F. Cheney gave an instance of a geometric proof of the law of tangents in trigonometry. In the first discussion below Professor Lovitt shows how a number of such proofs may be obtained. The two papers contain references to a number of other similar proofs.

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[1] The smallest positive value of β is approximately 12.94.